

# Multiple Schramm-Loewner evolutions for conformal field theories with Lie algebra symmetries

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## Abstract

We provide multiple Schramm-Loewner evolutions (SLE) to describe the scaling limit of multiple interfaces in critical lattice models possessing Lie algebra symmetries. The critical behavior of the models is described by Wess-Zumino-Witten (WZW) models. Introducing a multiple Brownian motion on a Lie group as well as that on the real line, we construct the multiple SLE with additional Lie algebra symmetries. The connection between the resultant SLE and the WZW model can be understood via SLE martingales satisfied by the correlation functions in the WZW model. Due to interactions among SLE traces, these Brownian motions have drift terms which are determined by partition functions for the corresponding WZW model. As a concrete example, we apply the formula to the  $\widehat{\mathfrak{su}}(2)_k$ -WZW model. Utilizing the fusion rules in the model, we conjecture that there exists a one-to-one correspondence between the partition functions and the topologically inequivalent configurations of the SLE traces. Furthermore, solving the Knizhnik-Zamolodchikov equation, we exactly compute the probabilities of occurrence for certain configurations (i.e. crossing probabilities) of traces for the triple SLE.

## 1 Introduction

Geometric aspects of critical phenomena are characterized by random fractals such as conformally invariant fluctuations of local order parameters. They have been extensively studied from various different points of view, especially in two dimensions (2D) where the conformal invariance imposes strong constraints on the structure of critical phenomena. Among them, the Schramm-Loewner evolutions (SLEs) [1], which directly describe geometric aspects of 2D critical phenomena through simple 1D Brownian motions, have brought a renewed interest in the theory of random fractals (see [2, 3, 4, 5, 6, 7, 8, 9] for reviews).

The SLE is a stochastic process defined in the upper half plane  $\mathbb{H}$ . Its evolution is described by the ordinary differential equation

$$dg_t(z) = \frac{2dt}{g_t(z) - x_t}, \quad g_0(z) = z \in \mathbb{H}, \quad (1.1)$$

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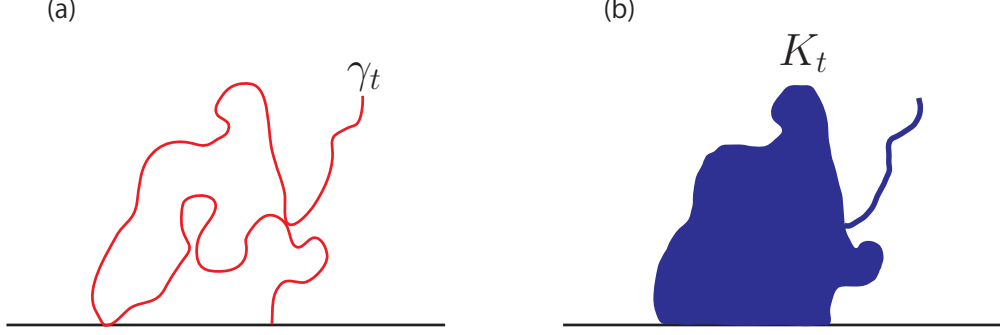


Figure 1: Schematic view of a SLE trace ( $\gamma_t$  denotes the tip of the trace) (a) and its hull  $K_t$  (b).

where  $x_t = \sqrt{\kappa}\xi_t$  is a Brownian motion on  $\mathbb{R}$ , starting at the origin (i.e.  $x_0 = 0$ ), and its expectation value and variance are given by  $\mathbf{E}[dx_t] = 0$  and  $\mathbf{E}[dx_t dx_t] = \kappa dt$ , respectively. Here  $\kappa > 0$  is a diffusion coefficient which essentially characterizes the SLE process. The SLE (1.1) has a solution up to the explosion time  $\tau_z$ , i.e. the first time when  $g_t(z)$  hits the singularity  $x_t$ . Let  $K_t = \{z \in \mathbb{H} | \tau_z < t\}$  be the hull at time  $t$  (see Fig. 1 for a schematic view). Then  $K_t$  ( $t \geq 0$ ) is an increasing family of hulls:  $K_s \subset K_t$  for  $s < t$ . Moreover  $g_t(z)$  with  $g_t(z) = z + O(1)$  at  $z \rightarrow \infty$  (hydrodynamic normalization) is the unique conformal map uniformizing the complement of the hull  $K_t$  in the upper half plane  $\mathbb{H}$ :  $g_t(z) : \mathbb{H} \setminus K_t \rightarrow \mathbb{H}$  (see Fig. 2). The image  $x_t$  by  $g_t^{-1}(x_t)$  defines the tip  $\gamma_t$  of the growing random curve. More precisely, it is expressed as  $\gamma_t = \lim_{\epsilon \rightarrow +0} g_t^{-1}(x_t + i\epsilon)$ .

The connection between the SLE (1.1) and conformal field theory (CFT) is well understood [10, 11, 12, 13, 14, 15, 16, 17, 5, 7]. Specifically, it can be accomplished by noticing that CFT correlation functions

$$\mathcal{M}_t = \frac{\langle \psi(\infty) \mathcal{O} \psi(\gamma_t) \rangle}{\langle \psi(\infty) \psi(\gamma_t) \rangle} \quad (1.2)$$

are SLE martingales:  $d\mathcal{M}_t/dt = 0$ , where  $\psi(\gamma_t)$  and  $\psi(\infty)$  are boundary condition changing (bcc) operators with conformal weights  $h$ , which are inserted at the points  $z = \gamma_t$  and  $z = \infty$ , respectively (see [7, 17] or next section for details). Thus one can find the SLE corresponds to the minimal conformal field theory  $\mathcal{M}(p, p')$  ( $p, p'$  are coprime integers satisfying the condition  $p > p' \geq 2$ ) where the central charge and the conformal weights of the primary fields are, respectively, given by [18, 19]

$$c = 1 - 6 \frac{(p - p')^2}{pp'}, \quad h_{r,s} = \frac{(pr - p's)^2 - (p - p')^2}{4pp'}. \quad (1.3)$$

Then the diffusion constant  $\kappa$  and the conformal weight  $h$  of the boundary field  $\psi$  are, respectively, expressed as

$$c = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}, \quad h = \frac{6 - \kappa}{2\kappa} = \begin{cases} h_{2,1} & \text{for } \kappa = 4p'/p \leq 4 \\ h_{1,2} & \text{for } \kappa = 4p/p' > 4 \end{cases}. \quad (1.4)$$

Namely the boundary field  $\psi$  is degenerate at level two, i.e. possesses a null field at level two.

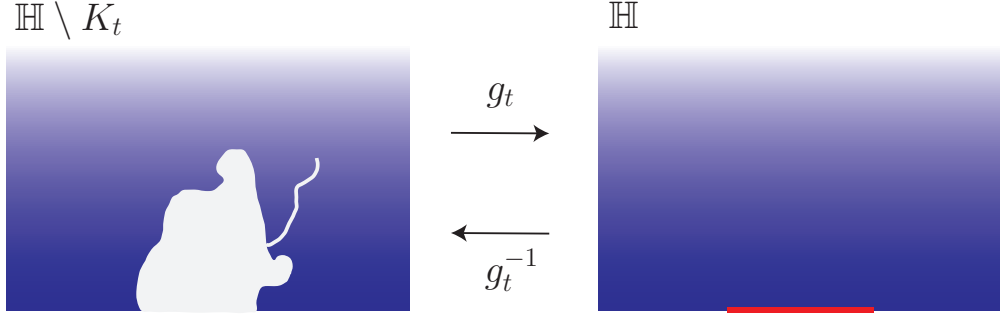


Figure 2: Uniformizing map  $g_t: \mathbb{H} \setminus K_t \rightarrow \mathbb{H}$  and its inverse.

An extension to the SLE connecting with conformal field theories with Lie algebra symmetries (i.e. WZW models [20, 21, 22]) has been achieved by adding the extra Brownian motion  $\exp(\sum_a t^a d\theta_t^a(z))$  on a (semisimple) group manifold  $G$  associated with a Lie algebra  $\mathfrak{g}$ , where  $t^a$ 's ( $a = 1, \dots, \dim \mathfrak{g}$ ) stand for any representation of the Lie algebra generators [23, 24] (see [25] for a very different approach). The evolution of this additional stochastic process  $\theta_t^a(z)$  is defined as

$$d\theta_t^a(z) = \frac{\sqrt{\tau} d\vartheta_t^a}{z - x_t}, \quad \mathbf{E}[d\vartheta_t^a] = 0, \quad \mathbf{E}[d\vartheta_t^a d\vartheta_t^b] = \delta^{ab} dt \quad (\vartheta_t^a \in \mathbb{R}). \quad (1.5)$$

The combination of (1.1) and (1.5) defines a fractal curve living on the Lie group  $G$  manifold. The SLE martingale  $d\mathcal{M}_t/dt = 0$ , which should be satisfied by the CFT correlation function (1.2) (note that the bcc operators and the operator  $\mathcal{O}$  take their values on  $G$ ), determines the relation of the SLE defined by (1.1) and (1.5) with the corresponding WZW model. Some extensions to the SLE with other additional symmetries have also been done in [26, 27, 28, 29, 30, 31].

In this paper we generalize the SLE for the system containing multiple random interfaces which possess additional Lie algebra symmetries, according to the theory developed in [17]. Assuming that each SLE interface grows under an independent martingale in the infinitesimal time interval, we extend (1.1) together with (1.5) to the case for the system with multiple interfaces. The evolution (cf. (1.1) for the single case) characterizing the geometric aspect of the interfaces is described by a multiple Brownian motion on the real line, while the evolution (cf. (1.5) for the single case) expressing the algebraic aspect is described by a multiple Brownian motion on the Lie group  $G$ . Both the Brownian motions, however, have drift terms describing the interaction among the SLE traces. Taking into account the SLE martingale, one finds that these drift terms are determined by the partition function in the corresponding WZW model. Moreover this partition function characterizes the configuration of the SLE traces. As an example, we apply our formula to the SLE for the  $\widehat{\mathfrak{su}}(2)_k$ -WZW models. Utilizing the fusion rules for  $\widehat{\mathfrak{su}}(2)_k$ , we conjecture that there is a one-to-one correspondence between the partition functions and the topologically inequivalent configurations of the SLE traces. Further, we exactly compute the probabilities of occurrence for certain configurations of traces (that correspond to crossing probabilities) for the triple SLE, which can be obtained by solving the Knizhnik-Zamolodchikov (KZ) equation [22].

The paper is organized as follows. In the subsequent section, we describe some basic

notions required in this paper. In section 3, the multiple SLE for conformal field theory with Lie algebra symmetries is formulated. The drift terms of the driving Brownian motions are explicitly determined in section 4. A concrete application of the formula to the SLE for the  $\widehat{\mathfrak{su}}(2)_k$ -WZW model is given in section 5.

## 2 Preliminaries

In this section, we describe several theoretical foundations required in subsequent sections. In the former part of this section, we introduce SLE martingales from the point of view of statistical mechanics (see [7] for details). SLE martingales are given by CFT correlation functions involving bcc operators, and hence they become a key to decipher the relation between SLE and CFT. In the latter part, general properties of correlation functions are explained in the case of the WZW model.

### 2.1 SLE martingales and conformal correlators

To formulate the SLEs correctly describing the behavior of 2D interfaces in critical systems, one must construct SLE martingales in terms of the corresponding statistical systems defined on  $\mathbb{H}$ . Let  $\langle \mathcal{O} \rangle$  be the thermal average of an observable  $\mathcal{O}$  defined in  $\mathbb{H}$ , and  $\langle \mathcal{O} \rangle|_{\{\gamma_t\}}$  be the thermal average under a given shape of configuration of (multiple) interfaces, where  $\{\gamma_t\}$  denotes the shape of configuration with its occurrence probability  $\mathbf{P}[\{\gamma_t\}]$ . Then the thermal average  $\langle \mathcal{O} \rangle$  must be given by

$$\langle \mathcal{O} \rangle = \mathbf{E}[\langle \mathcal{O} \rangle|_{\{\gamma_t\}}] = \sum_{\{\gamma_t\}} \mathbf{P}[\{\gamma_t\}] \langle \mathcal{O} \rangle|_{\{\gamma_t\}}, \quad (2.1)$$

where the average  $\mathbf{E}[\cdots]$  should be taken over all the possible configurations. The conditional expectation value  $\langle \mathcal{O} \rangle|_{\{\gamma_t\}}$  is thus time independent (i.e. conserved in mean), and therefore it is a martingale. Here and in what follows, we denote it by  $\mathcal{M}_t$ .

At the critical point where the conformal invariance is expected in the system, the above observable can be described in terms of the CFT correlation functions. For the situation where the number of the interfaces under consideration is  $m$ , it reads

$$\mathcal{M}_t := \langle \mathcal{O} \rangle|_{\{\gamma_t\}} = \frac{\langle \mathcal{O} \psi_1(w_1) \cdots \psi_m(w_m) \psi_{m+1}(\infty) \rangle}{\langle \psi_1(w_1) \cdots \psi_m(w_m) \psi_{m+1}(\infty) \rangle}, \quad (2.2)$$

where  $w_j$ 's denote the positions of the tips of the interfaces. Note that the CFT correlation functions are defined on the domain  $\mathbb{H}$  removing the hull  $K_t$ , i.e.  $\mathbb{H} \setminus K_t$  (see Fig. 3). The operators  $\psi_j(w_j)$ 's inserted at the positions of the tips denote primary bcc operators with conformal weights  $h$ : under a local conformal map  $z \rightarrow z' = w(z)$ , a primary field  $\psi(z)$  transforms as  $\psi(z) \rightarrow \psi'(z')$ :

$$\psi'(z') = \left( \frac{dw(z)}{dz} \right)^{-h} \psi(z). \quad (2.3)$$

The denominator in (2.2) stands for the CFT partition function with a specific boundary condition fixed by the bcc operators  $\psi_j(w_j)$ .

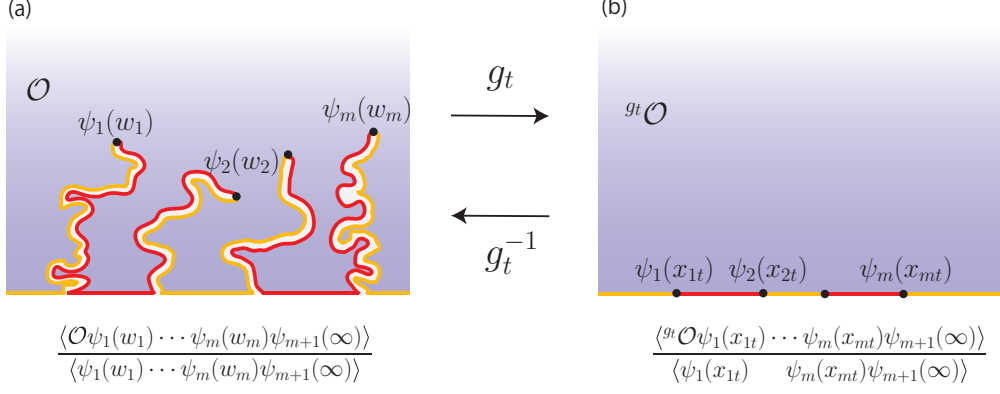


Figure 3: CFT correlation functions on  $\mathbb{H} \setminus \mathbb{K}$  (a). It can be transformed to the one defined on  $\overline{\mathbb{H}}$  by the uniformizing map  $g_t(z)$  (b).

Applying the conformal uniformizing map  $g_t(z)$  (written with the same symbol as that used for the single SLE (1.1)), we obtain

$$\mathcal{M}_t = \frac{\langle {}^{g_t}\mathcal{O} \psi_1(x_{1t}) \cdots \psi_m(x_{mt}) \psi_{m+1}(\infty) \rangle}{\langle \psi_1(x_{1t}) \cdots \psi_m(x_{mt}) \psi_{m+1}(\infty) \rangle}. \quad (2.4)$$

Here  $x_{jt} = g_t(w_j)$ , and  ${}^{g_t}\mathcal{O}$  is the image of  $\mathcal{O}$  by the map  $g_t$ . Note that the Jacobians coming from the conformal map on  $\psi_j$  have been canceled in the numerators and the denominators<sup>1</sup>. Now SLE martingales  $\mathcal{M}_t$  are expressed as the CFT correlation functions on  $\overline{\mathbb{H}}$ , where the bcc operators  $\psi_j(x_j)$  are inserted at the points  $x_j \in \mathbb{R}$  (see Fig. 3).

To proceed further, let us consider the case when the operator  $\mathcal{O}$  is a product of an arbitrary number of primary fields  $\mathcal{O} = \prod_{j=1}^n \phi_j(z_j, \bar{z}_j)$  at positions  $(z_j, \bar{z}_j)$  and with conformal weights  $(h_j, \bar{h}_j)$ . By construction, the uniformizing map  $g_t(z)$  can be analytically extended to the lower half plane:  $g_t(\bar{z}) = \bar{g}_t(z)$ . Then, the doubling trick can be applied to the CFT correlation functions. The result reads

$$\mathcal{M}_t = \prod_{j=1}^{2n} \left( \frac{\partial y_{jt}}{\partial z_j} \right)^{h_j} \frac{\langle \prod_{j=1}^{2n} \phi_j(y_{jt}) \psi_1(x_{1t}) \cdots \psi_m(x_{mt}) \psi_{m+1}(\infty) \rangle}{\langle \psi_1(x_{1t}) \cdots \psi_m(x_{mt}) \psi_{m+1}(\infty) \rangle}, \quad (2.5)$$

where we denote that  $y_{jt} = g_t(z_j)$ ;  $\phi_{j+n} = \bar{\phi}_j$ ;  $z_{j+N} = \bar{z}_j = z_j^*$ ;  $y_{j+Nt} = \bar{y}_{jt} = y_{jt}^*$ ,  $h_{j+N} = \bar{h}_j$  for  $1 \leq j \leq n$ . Here  $\phi_j(y_{jt})$  ( $\bar{\phi}_j(y_{jt})$ ) stands for the holomorphic (antiholomorphic) part of the field  $\phi_j(y_{jt}, \bar{y}_{jt})$ .

In this paper, we analyze the SLE martingales (2.5) for the system that possesses additional Lie algebra symmetries. Namely we construct the multiple SLE for the WZW models which are one of the most fundamental CFTs with extra Lie algebra symmetries. In this case, the primary fields constructing the SLE martingale (2.5) possess internal degrees of freedom, such as “spin”.

<sup>1</sup> The identity  $\langle \Phi'_1(z'_1) \cdots \Phi'_n(z'_n) \rangle = \langle \Phi_1(z'_1) \cdots \Phi_n(z'_n) \rangle$  holds for a global conformal map  $z \rightarrow z' = f(z)$ , where the field  $\Phi_j(z_j)$  transforms as  $\Phi_j(z_j) \rightarrow \Phi'_j(z'_j)$  [19].

## 2.2 WZW models and correlation functions

Let us introduce several properties for correlation functions of WZW primary fields to analyze the SLE martingales (2.5). The WZW model is a CFT described by a field  $g(z, \bar{z})$  taking values in a group manifold  $G$  associated with a Lie algebra  $\mathfrak{g}$  [19, 20, 21, 22]. The model is invariant under

$$g(z, \bar{z}) \rightarrow \Omega(z)g(z, \bar{z})\bar{\Omega}^{-1}(\bar{z}), \quad (2.6)$$

where  $\Omega$  and  $\bar{\Omega}(\bar{z})$  denote arbitrary matrices valued in  $G$ . This invariance gives rise to Noether currents  $J(z) = -k\partial_z g g^{-1}$  and  $\bar{J}(\bar{z}) = k g^{-1}\partial_{\bar{z}} g$ , which can be written as

$$J(z) = \sum_{a=1}^{\dim \mathfrak{g}} J^a(z)t^a, \quad \bar{J}(\bar{z}) = \sum_{a=1}^{\dim \mathfrak{g}} \bar{J}^a(\bar{z})t^a, \quad (2.7)$$

where  $t^a$ 's stand for any matrix representation of the generator of  $\mathfrak{g}$ , with commutation relations  $[t^a, t^b] = \sum_c i f^{ab}_c t^c$ . The parameter  $k$  is a positive integer referred to as the level. Hereafter we only consider the holomorphic components, as if there were no boundary (cf. (2.5)) [32]. Let  $\langle X \rangle$  be a correlator of  $G$ -valued fields. Then the infinitesimal transformation  $\Omega(z) = 1 + \omega(z)$  leads to the Ward identity

$$\delta_\omega \langle X \rangle = -\frac{1}{2\pi i} \oint dz \sum_{a=1}^{\dim \mathfrak{g}} \omega^a(z) \langle J^a(z) X \rangle. \quad (2.8)$$

On the other hand, conformal aspects of the WZW model are described by the stress energy tensor  $T(z)$ :

$$T(z) = \frac{1}{2(k + h^\vee)} \sum_{a=1}^{\dim \mathfrak{g}} : J^a(z) J^a(z) :, \quad (2.9)$$

where  $h^\vee$  denotes the dual Coxeter number of  $\mathfrak{g}$ . Note that  $: :$  means the “normal ordering” defined as

$$: A(z) B(z) := \frac{1}{2\pi i} \oint_z \frac{dw}{w - z} A(w) B(z). \quad (2.10)$$

The infinitesimal conformal transformation  $z \rightarrow z' = z + \epsilon(z)$  leads to the Ward identity

$$\delta_\epsilon \langle X \rangle = -\frac{1}{2\pi i} \oint dz \epsilon(z) \langle T(z) X \rangle. \quad (2.11)$$

This geometric part of the Ward identity (2.11) together with the algebraic part (2.8) are a key ingredient in analyzing the SLE martingale.

The current  $J^a(z)$  and the stress energy tensor  $T(z)$  can be expanded in terms of the modes  $J_n^a$  and  $L_n$ , respectively. Namely

$$J^a(z) = \sum_{n \in \mathbb{Z}} J_n^a z^{-n-1}, \quad T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}, \quad (2.12)$$

where  $J_n^a$  and  $L_n$  denote, respectively, the generators<sup>2</sup> of the affine Lie algebra  $\hat{\mathfrak{g}}$  with level  $k$ , and the generators of the Virasoro algebra:

$$\begin{aligned} [J_n^a, J_m^b] &= \sum_c i f_{ab}^c J_{n+m}^c + kn \delta^{a,b} \delta_{n+m,0}, \\ [L_n, J_m^a] &= -m J_{n+m}^a, \\ [L_n, L_m] &= (n-m) L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n+m,0}. \end{aligned} \quad (2.13)$$

The central charge  $c$  of the Virasoro algebra is then given by

$$c = \frac{k \dim \mathfrak{g}}{k + h^\vee}. \quad (2.14)$$

By construction (2.9), the Virasoro generators are not independent from the affine generators:

$$L_n = \frac{1}{2(k + h^\vee)} \sum_{a=1}^{\dim \mathfrak{g}} \sum_{m \in \mathbb{Z}} : J_m^a J_{n-m}^a :, \quad (2.15)$$

where the normal ordering  $: :$  means that the operator with larger index  $n$  is placed at the rightmost position.

The primary fields in the WZW model are defined as the fields transforming covariantly with respect to the  $G(z)$  transformation (2.6):  $\delta_\omega g = \omega g$ . Together with the conformal covariance (see (2.3)), these properties can be expressed in terms of the operator product expansions (OPE) via the Ward identities (2.11) and (2.8):

$$T(z) \psi_\lambda(w) = \frac{h_\lambda \psi_\lambda(w)}{(z-w)^2} + \frac{\partial_w \psi_\lambda(w)}{z-w} + \text{reg.}, \quad J^a(z) \psi_\lambda(w) = \frac{-t_\lambda^a \psi_\lambda(w)}{z-w} + \text{reg.}, \quad (2.16)$$

where the field  $\psi_\lambda(z)$  takes values in the representation specified by the highest weight  $\lambda$ , and  $t_\lambda^a$  is the generator  $t^a$  in that representation. Furthermore utilizing the field-state correspondence, i.e.  $|\psi_\lambda\rangle := \lim_{z \rightarrow 0} \psi_\lambda(z)|0\rangle$ , we can translate these properties into

$$\begin{aligned} L_0 |\psi_\lambda\rangle &= h_\lambda |\psi_\lambda\rangle, & L_n |\psi_\lambda\rangle &= 0 \quad (n > 0), \\ J_0^a |\psi_\lambda\rangle &= -t_\lambda^a |\psi_\lambda\rangle, & J_n^a |\psi_\lambda\rangle &= 0 \quad (n > 0). \end{aligned} \quad (2.17)$$

Insertion of the relation (2.15) into the l.h.s of the first equation in the above yields

$$L_0 |\psi_\lambda\rangle = \frac{1}{2(k + h^\vee)} \sum_{a=1}^{\dim \mathfrak{g}} J_0^a J_0^a |\psi_\lambda\rangle = \frac{1}{2(k + h^\vee)} \sum_{a=1}^{\dim \mathfrak{g}} t_\lambda^a t_\lambda^a |\psi_\lambda\rangle = \frac{(\lambda, \lambda + 2\rho)}{2(k + h^\vee)} |\psi_\lambda\rangle. \quad (2.18)$$

In the last equality we used the explicit form of the eigenvalue of the quadratic Casimir. The quantity  $\rho$  denotes the Weyl vector, i.e. the sum of the fundamental weights  $\rho = \sum_{i=1}^r \Lambda_i$ , where  $r$  is the rank of  $\mathfrak{g}$ . Comparing the r.h.s in the first equation in (2.17) with the above, we find

$$h_\lambda = \frac{(\lambda, \lambda + 2\rho)}{2(k + h^\vee)}. \quad (2.19)$$

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<sup>2</sup>Here we use the orthonormal basis in terms of the Killing form:  $K(J_n^a, J_m^b) = \delta_{n+m,0} \delta^{a,b}$ . In that case, the structure constants can be written as  $f_{ab}^c = f_{abc}$ , where  $f_{abc}$  is antisymmetric in all three indices.

Since the Virasoro generators are expressed in terms of affine generators as (2.15), the other arbitrary states are of the form  $J_{-l_1}^a J_{-l_2}^b \cdots |\psi_\lambda\rangle$  with  $l_1, l_2 \dots$  positive integers. Let  $(L_{-l}\psi_\lambda)(z)$  and  $(J_{-l}^a\psi_\lambda)(z)$  be the descendent fields corresponding to the states  $L_{-l}|\psi_\lambda\rangle$  and  $J_{-l}^a|\psi_\lambda\rangle$ , respectively. Combining the OPE (2.16) and the mode expansions (2.12), one finds that the correlation functions  $\langle(L_{-l}\psi_\lambda)(z)X\rangle$  and  $\langle(J_{-l}^a\psi_\lambda)(z)X\rangle$ , where  $X = \prod_{i=1}^m \psi_{\lambda_i}(z_i)$ , satisfy the following equations:

$$\begin{aligned}\langle(L_{-l}\psi_\lambda)(z)X\rangle &= \mathcal{L}_{-l}\langle\psi_\lambda(z)X\rangle, \quad \mathcal{L}_{-l} = \sum_{j=1}^m \left[ \frac{(l-1)h_{\lambda_j}}{(z_j - z)^l} - \frac{\partial_{z_j}}{(z_j - z)^{l-1}} \right], \\ \langle(J_{-l}^a\psi_\lambda)(z)X\rangle &= \mathcal{J}_{-l}^a\langle\psi_\lambda(z)X\rangle, \quad \mathcal{J}_{-l}^a = \sum_{j=1}^m \frac{t_{\lambda_j}^a}{(z_j - z)^l}.\end{aligned}\tag{2.20}$$

Note that the global  $G$ -invariance requires  $\delta_\omega\langle\psi_\lambda(z)X\rangle = 0$ . Using the Ward identity (2.8) with constant  $\omega$  and the OPE (2.16), we obtain a relation satisfied by the correlation function:

$$\sum_{a=1}^{\dim \mathfrak{g}} \left( t_\lambda^a + \sum_{j=1}^m t_{\lambda_j}^a \right) \langle\psi_\lambda(z)X\rangle = 0.\tag{2.21}$$

This constraint together with the global conformal invariance (or equivalently  $SL(2, \mathbb{C})$ -invariance) fix the structure of the two- and three-points correlation functions.

To close this section, let us derive a crucial equation called Knizhnik-Zamolodchikov (KZ) equation [22] satisfied by the correlation functions of the WZW primary fields. The constraint stems from the fact that the Virasoro generators are not independent from the affine generators as in (2.15). For  $n = -1$ , we have  $L_{-1} = \sum_a J_{-1}^a J_0^a / (k + h^\vee)$ . Then a null state  $|\chi\rangle = 0$  is given by

$$|\chi\rangle = \left( L_{-1} + \frac{1}{k + h^\vee} \sum_{a=1}^{\dim \mathfrak{g}} t_\lambda^a J_{-1}^a \right) |\psi_\lambda\rangle = 0,\tag{2.22}$$

where we used the property in (2.17). Using the field-state correspondence and inserting the property (2.20) into the correlation functions  $\langle\chi(z)X\rangle$ , one obtains the KZ equation:

$$\left( \partial_z - \frac{1}{k + h^\vee} \sum_{a=1}^{\dim \mathfrak{g}} \sum_{j=1}^m \frac{t_\lambda^a t_{\lambda_j}^a}{z - z_j} \right) \langle\chi(z)X\rangle = 0.\tag{2.23}$$

Here we used the translation invariance  $(\partial_z + \sum_j \partial_{z_j})\langle\chi(z)X\rangle = 0$ , which can be easily verified from (2.11) and the OPE (2.16) by setting  $\epsilon(z) = \epsilon$ .

### 3 Multiple SLEs for WZW models

Now we generalize the SLE (1.1) and (1.5) for the system containing multiple random interfaces with additional Lie algebra symmetries. In the infinitesimal time interval, we expect that each SLE interface grows under an independent martingale. Let us discuss the case where



the number of the interfaces is  $m$ . Then the uniformizing map  $g_t(z)$  with the hydrodynamic normalization may be of the form [17] (see also [33, 34, 35] for other approaches).

$$dg_t(z) = \sum_{\alpha=1}^m \frac{2dq_\alpha}{g_t(z) - x_{\alpha t}}, \quad g_0(z) = z, \quad (3.1)$$

where  $dq_\alpha$ 's mean infinitesimal time intervals satisfying the condition  $\sum_{\alpha=1}^m dq_\alpha = dt$ . The random processes  $x_\alpha$  ( $1 \leq \alpha \leq m$ ), which play a role as driving forces for the growth of interfaces, should be written as the Itô stochastic differential equations:

$$dx_{\alpha t} = \sqrt{\kappa} d\xi_{\alpha t} + dF_{\alpha t}, \quad (3.2)$$

where  $\xi_{\alpha t}$  is an  $\mathbb{R}^m$ -valued Brownian motion whose expectation value and variance are, respectively, given by

$$\mathbf{E}[d\xi_{\alpha t}] = 0, \quad \mathbf{E}[d\xi_{\alpha t} d\xi_{\beta t}] = \delta_{\alpha\beta} dq_\alpha. \quad (3.3)$$

Namely  $dq_\alpha$  prescribes the growth rate of each interface. The quantity  $dF_{\alpha t}$  denotes a drift term proportional to  $dq_\alpha$ , which comes from interactions among interfaces, and will be determined later by the SLE martingale.

To extend (3.1) to evolutions with Lie algebra symmetries, we define a stochastic process  $\exp[\sum_a d\theta_t^a(z)t^a]$  ( $a = 1, \dots, \dim \mathfrak{g}$ ) living on a Lie group manifold  $G(z)$ , where  $\theta_t^a(z)$  is written as

$$d\theta_t^a(z) = \sum_{\alpha=1}^m \frac{dp_{\alpha t}^a}{z - x_{\alpha t}}, \quad \theta_0^a(z) = 0 \quad (3.4)$$

with

$$dp_{\alpha t}^a = \sqrt{\tau} d\vartheta_{\alpha t}^a + dG_{\alpha t}^a. \quad (3.5)$$

Here  $\vartheta_{\alpha t}^a$  is an  $\mathbb{R}^{m \dim \mathfrak{g}}$ -valued Brownian motion with

$$\mathbf{E}[d\vartheta_{\alpha t}^a] = 0, \quad \mathbf{E}[d\vartheta_{\alpha t}^a d\vartheta_{\beta t}^b] = \delta^{ab} \delta_{\alpha\beta} dq_\alpha, \quad (3.6)$$

and  $dG_{\alpha t}^a$  stands for a drift term proportional to  $dq_\alpha$ , which will be also determined later. For  $z = x_{\beta t}$ , we must define

$$d\theta_t^a(x_{\beta t}) = \sum_{\substack{\alpha=1 \\ \alpha \neq \beta}}^m \frac{dp_{\alpha t}^a}{x_{\beta t} - x_{\alpha t}}. \quad (3.7)$$

The growth of interfaces with affine Lie algebra symmetries is described by both the geometric (3.1) and the algebraic (3.4) components.

## 4 Drift terms and SLE martingales

The remaining problem is to determine the drift terms appearing in the driving forces (3.2) and in the Brownian motion (3.5). To achieve it, we must evaluate the variation of the SLE martingale  $\mathcal{M}_t$  involving WZW primary fields. For the WZW models, the SLE martingale is written as

$$\mathcal{M}_t = \prod_{j=1}^{2n} \left( \frac{\partial y_{jt}}{\partial z_j} \right)^{h_{\mu_j}} \frac{\langle \prod_{j=1}^{2n} \phi_{\mu_j}(y_{jt}) \psi_{\lambda_1}(x_{1t}) \cdots \psi_{\lambda_m}(x_{mt}) \psi_{\lambda_{m+1}}(\infty) \rangle}{\langle \psi_{\lambda_1}(x_{1t}) \cdots \psi_{\lambda_m}(x_{mt}) \psi_{\lambda_{m+1}}(\infty) \rangle}, \quad (4.1)$$

where  $y_{jt} = g_t(z_j)$  and  $x_{\alpha t} = g_t(w_\alpha)$  (see section 2.1 for details). Note that the primary fields  $\phi_{\mu_j}$  and  $\psi_{\lambda_\alpha}$  constructing the correlation function take values in the representation specified by the highest weights  $\mu_j$  and  $\lambda_\alpha$ , respectively.

The drift terms are determined by the condition which makes  $\mathcal{M}_t$  to be a martingale. To simplify the notations we sometimes omit the index  $t$ ,  $\lambda$  and  $\mu$  (e.g.  $x_{\alpha t} = x_\alpha$ ,  $\theta_t^a = \theta^a$ ,  $\psi_{\lambda_\alpha}(x_\alpha) = \psi_\alpha(x_\alpha)$ ,  $h_{\lambda_j} = h_j$ , etc.), if there is no ambiguity. Let  $Z_t$ ,  $Z_t^\phi$  and  $J_t^\phi$  be the denominator, numerator and Jacobian factor of (4.1), respectively. First we explicitly compute  $dJ_t^\phi$ . A simple manipulation leads to

$$\partial_{q_\alpha} \left( \frac{\partial y}{\partial z} \right)^h = h \left( \frac{\partial y}{\partial z} \right)^{h-1} \partial_z \left( \frac{\partial y}{\partial q_\alpha} \right) = h \left( \frac{\partial y}{\partial z} \right)^h \partial_y \left( \frac{\partial y}{\partial q_\alpha} \right) = - \left( \frac{\partial y}{\partial z} \right)^h \frac{2h}{(y - x_\alpha)^2}, \quad (4.2)$$

where in the last equality, we applied the geometric component of the SLE (3.1). Thus we obtain the variation of the Jacobian factor:

$$dJ_t^\phi = -J_t^\phi \sum_{\alpha=1}^m \sum_{j=1}^{2n} \frac{2h_j}{(y_j - x_\alpha)^2} dq_\alpha. \quad (4.3)$$

Next we shall calculate  $d(J_t^\phi Z_t^\phi)$ . The Itô derivative of the bulk fields  $\phi(y)$  is given by

$$\begin{aligned} d\phi(y) &= \frac{\partial \phi(y)}{\partial y} \sum_{\alpha=1}^m \frac{\partial y}{\partial q_\alpha} dq_\alpha + \sum_{a=1}^{\dim \mathfrak{g}} t^a d\theta^a(y) \phi(y) + \frac{1}{2} \sum_{a=1}^{\dim \mathfrak{g}} t^a d\theta^a(y) \sum_{b=1}^{\dim \mathfrak{g}} t^b d\theta^b(y) \phi(y) \\ &= \sum_{\alpha=1}^m \left[ \frac{2dq_\alpha \partial_y}{y - x_\alpha} + \sum_{a=1}^{\dim \mathfrak{g}} \frac{dp_\alpha^a t^a}{y - x_\alpha} + \frac{\tau}{2} \sum_{a=1}^{\dim \mathfrak{g}} \frac{dq_\alpha t^a t^a}{(y - x_\alpha)^2} \right] \phi(y). \end{aligned} \quad (4.4)$$

Here we applied (3.1) to the first term, and (3.4) to the second and third terms. For the third term we also used the property (3.6). Similarly, for the boundary fields  $\psi_\alpha(x_\alpha)$ , we obtain

$$\begin{aligned} d\psi_\alpha(x_\alpha) &= \left[ dq_\alpha \frac{\kappa}{2} \partial_{x_\alpha}^2 + dx_\alpha \partial_{x_\alpha} + \sum_{a=1}^{\dim \mathfrak{g}} t^a d\theta^a(x_\alpha) + \frac{1}{2} \sum_{a=1}^{\dim \mathfrak{g}} t^a d\theta^a(x_\alpha) \sum_{b=1}^{\dim \mathfrak{g}} t^b d\theta^b(x_\alpha) \right] \psi_\alpha(x_\alpha) \\ &= \left[ dq_\alpha \frac{\kappa}{2} \partial_{x_\alpha}^2 + dx_\alpha \partial_{x_\alpha} + \sum_{a=1}^{\dim \mathfrak{g}} \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^m \left( \frac{dp_\beta^a t_\alpha^a}{x_\alpha - x_\beta} + \frac{\tau}{2} \frac{dq_\beta t_\alpha^a t_\alpha^a}{(x_\alpha - x_\beta)^2} \right) \right] \psi_\alpha(x_\alpha). \end{aligned} \quad (4.5)$$

The above two relations together with (4.3) give the form of  $d(J_t^\phi Z_t^\phi)$ . Explicitly it reads

$$\begin{aligned} \frac{d(J_t^\phi Z_t^\phi)}{J_t^\phi} &= \sum_{\alpha=1}^m dq_\alpha \left( \frac{\kappa}{2} \mathcal{L}_{\alpha,-1}^2 - 2\mathcal{L}_{\alpha,-2} + \frac{\tau}{2} \sum_{a=1}^{\dim \mathfrak{g}} \mathcal{J}_{\alpha,-1}^a \mathcal{J}_{\alpha,-1}^a \right) Z_t^\phi \\ &\quad + \sum_{\alpha=1}^m \left( 2\tilde{\mathcal{L}}_{\alpha,-2} dq_\alpha + \mathcal{L}_{\alpha,-1} dx_\alpha + \sum_{a=1}^{\dim \mathfrak{g}} \mathcal{J}_{\alpha,-1}^a dp_\alpha^a \right) Z_t^\phi, \end{aligned} \quad (4.6)$$

where we define

$$\begin{aligned}\mathcal{L}_{\alpha,-l} &= \sum_{j=1}^n \left( \frac{(l-1)h_j}{(y_j - x_\alpha)^l} - \frac{\partial_{y_j}}{(y_j - x_\alpha)^{l-1}} \right) + \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^m \left( \frac{(l-1)h_\beta}{(x_\beta - x_\alpha)^l} - \frac{\partial_{x_\beta}}{(x_\beta - x_\alpha)^{l-1}} \right), \\ \mathcal{J}_{\alpha,-l}^a &= \sum_{j=1}^n \frac{t_{\lambda_j}^a}{(y_j - x_\alpha)^l} + \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^m \frac{t_{\lambda_\beta}^a}{(x_\beta - x_\alpha)^l},\end{aligned}\tag{4.7}$$

and  $\tilde{\mathcal{L}}$  (resp.  $\tilde{\mathcal{J}}$ ) is given by subtracting the first sum depending on  $y_j$  from the r.h.s of the first (resp. second) equation in (4.7). Note that the operator  $\mathcal{L}$  (resp.  $\mathcal{J}$ ) characteristically appears in the correlation functions between a descendent field  $(L_{-l}\psi_{\lambda_\alpha})(x_\alpha)$  (resp.  $(J_{-l}\psi_{\lambda_\alpha})(x_\alpha)$ ) and some composite primary field (see (2.20)). To derive (4.6), we have also applied the identity  $\partial_{x_\alpha} Z_t^\phi = \mathcal{L}_{\alpha,-1} Z_t^\phi$ , which comes from the translation invariance of the correlation functions (see the explanation below (2.23)).

In completely the same manner,  $dZ_t$  can also be evaluated:

$$\begin{aligned}dZ_t &= \sum_{\alpha=1}^m dq_\alpha \left( \frac{\kappa}{2} \tilde{\mathcal{L}}_{\alpha,-1}^2 - 2\tilde{\mathcal{L}}_{\alpha,-2} + \frac{\tau}{2} \sum_{a=1}^{\dim \mathfrak{g}} \tilde{\mathcal{J}}_{\alpha,-1}^a \tilde{\mathcal{J}}_{\alpha,-1}^a \right) Z_t \\ &\quad + \sum_{\alpha=1}^m \left( 2\tilde{\mathcal{L}}_{\alpha,-2} dq_\alpha + \tilde{\mathcal{L}}_{\alpha,-1} dx_\alpha + \sum_{a=1}^{\dim \mathfrak{g}} \tilde{\mathcal{J}}_{\alpha,-1}^a dp_\alpha^a \right) Z_t,\end{aligned}\tag{4.8}$$

By construction, we must set  $h_{\lambda_\alpha} = h$  ( $1 \leq \alpha \leq m$ ) for the conformal weights of the bcc operators. Furthermore, due to the constraint between the conformal weight  $h_\lambda$  and the representation  $\lambda$  (2.19), all the bcc operators  $\psi_{\lambda_\alpha}(x_\alpha)$  except for  $\psi_{\lambda_{m+1}}(\infty)$  must take values in the representation specified by  $\lambda$ , or its conjugate  $\lambda^*$  (note that  $h_\lambda = h_{\lambda^*}$  holds). The conformal weight  $h_{\lambda_{m+1}}$  for the bcc operator inserted at  $\infty$  is determined by fusion rules. Thanks to this restriction, we can determine, in principle, the relation of the parameters  $\kappa$ ,  $\tau$  and the weight  $h_\lambda$  of the bcc operators by considering the case that there is only a single interface ( $m = 1$ ) as in [23, 24]. We see this in the next subsection.

#### 4.1 Single case ( $m = 1$ )

For the single SLE (set  $\alpha = 1$  and  $m = 1$ ), one sees  $dZ_t = 0$  due to the translation invariance of the two-point correlation functions (it can also be obtained directly from (4.8) by setting  $m = 1$ ). Therefore the SLE martingale gives a constraint to the correlation function  $Z_t^\phi$ , i.e.  $\mathbf{E}[d(J_t^\phi Z_t^\phi)] = 0$ . From the relation (4.6) and the definitions (3.2) and (3.5), one easily see that this restriction leads to  $dF_1 = 0$ ,  $dG_1^a = 0$  and

$$\left( \frac{\kappa}{2} \mathcal{L}_{\alpha,-1}^2 - 2\mathcal{L}_{\alpha,-2} + \frac{\tau}{2} \sum_{a=1}^{\dim \mathfrak{g}} \mathcal{J}_{\alpha,-1}^a \mathcal{J}_{\alpha,-1}^a \right) Z_t^\phi = 0 \quad (\alpha = 1).\tag{4.9}$$

Namely any drift terms do not show up in the driving forces (3.2) and (3.5), as expected. Moreover the constraint (4.9) indicates that the bcc operators must have null states at level

2. Namely  $\mathcal{M}_t$  is a martingale if and only if the bcc operators have the following null states at level 2:

$$0 = |\chi\rangle := \left( \frac{\kappa}{2} L_{-1}^2 - 2L_{-2} + \frac{\tau}{2} \sum_{a=1}^{\dim \mathfrak{g}} J_{-1}^a J_{-1}^a \right) |\psi_{\lambda_1}\rangle. \quad (4.10)$$

Here  $L_n$  and  $J_n^a$  are, respectively, the Virasoro and affine generators (2.13). This is equivalent to the conditions  $J_1^b |\chi\rangle = 0$  and  $J_2^b |\chi\rangle = 0$ , which respectively lead to

$$\left[ (\tau k - \tau h^\vee - 2) J_{-1}^b + \kappa J_0^b L_{-1} + i\tau \sum_{a,c} f^{ab} J_0^a J_{-1}^c \right] |\phi_{\lambda_1}\rangle = 0, \quad (\kappa + \tau h^\vee - 4) J_0^b |\phi_{\lambda_1}\rangle = 0. \quad (4.11)$$

This necessary and sufficient condition is rather involved. For some simple cases (e.g.  $\widehat{\mathfrak{su}}(2)_k$ ), however, we can directly solve the above equations by acting the generator  $J_1^d$ . (Note that more elegant procedure utilizing the KZ-equation has been developed in [24]).

For later convenience, here we explicitly write down the results for the  $\widehat{\mathfrak{su}}(2)_k$ -WZW model. The central charge  $c$  (2.14) and the conformal weight  $h_{\lambda_1}$  (2.19) of the bcc operator  $\psi_{\lambda_1}(x_1)$  in the spin- $\lambda/2$  representation are, respectively, written as

$$c = \frac{3k}{k+2}, \quad h_{\lambda_1} = h_{\Lambda} = \frac{\lambda(\lambda+2)}{4(k+2)}, \quad (4.12)$$

where  $\Lambda$  denotes the fundamental weight, and we used  $h^\vee = 2$ . From the direct evaluation of (4.11) for  $\widehat{\mathfrak{su}}(2)_k$  case, one finds that the conditions in (4.11) are valid only for the case that the bcc operator carries spin-1/2 ( $\lambda = 1$ ) [24]. Then [23, 24]

$$\kappa = \frac{4(k+2)}{k+3}, \quad \tau = \frac{2}{k+3} \quad (k \geq 2). \quad (4.13)$$

For  $k = 1$ ,  $\kappa$  and  $\tau$  can not be specified, and only the relation  $\kappa + 2\tau = 4$  is imposed. This case, however, corresponds to a  $c = 1$  CFT (cf. (1.4)), and therefore we shall set  $\kappa = 4$ .

## 4.2 Multiple case ( $m > 1$ )

Now we identified the bcc operators  $\psi_{\lambda_j}(z_j)$ . Namely they have null states at level 2 and must satisfy the condition (4.10). Utilizing the field-state correspondence, one finds that the first sums in (4.6) and (4.8) vanish.

Thus the drift term of the Itô derivative of the CFT correlation function

$$d\mathcal{M}_t = d \left( \frac{J_t^\phi Z_t^\phi}{Z_t} \right) = \frac{d(J_t^\phi Z_t^\phi)}{Z_t} - \frac{J_t^\phi Z_t^\phi dZ_t}{Z_t^2} - \frac{d(J_t^\phi Z_t^\phi) dZ_t}{Z_t^2} + \frac{J_t^\phi Z_t^\phi (dZ_t)^2}{Z_t^3} \quad (4.14)$$

is explicitly given by

$$\begin{aligned} \mathbf{E}[d\mathcal{M}_t] = & J_t^\phi \sum_{\alpha=1}^m \left( dF_\alpha - \kappa dq_\alpha \partial_{x_\alpha} \log Z_t - 2 \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^m \frac{dq_\beta}{x_\alpha - x_\beta} \right) \partial_{x_\alpha} \left( \frac{Z_t^\phi}{Z_t} \right) \\ & + \frac{J_t^\phi}{Z_t} \sum_{\alpha=1}^m \sum_{a=1}^{\dim \mathfrak{g}} \left[ \left( dG_\alpha^a - \frac{\tau dq_\alpha}{Z_t} \tilde{J}_{\alpha,-1}^a Z_t \right) \left( J_{\alpha,-1}^a Z_t^\phi - \frac{Z_t^\phi \tilde{J}_{\alpha,-1}^a}{Z_t} Z_t \right) \right], \quad (4.15) \end{aligned}$$

where we substituted the relations (3.2) and (3.5). By recalling that  $\mathcal{M}_t$  is the SLE martingale, the above quantity must be zero. Thus one finds the drift terms  $dF_\alpha$  and  $dG_\alpha^a$  are described by the partition function  $Z_t$ :

$$dF_\alpha = \kappa dq_\alpha \partial_{x_\alpha} \log Z_t + 2 \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^m \frac{dq_\beta}{x_\alpha - x_\beta}, \quad dG_\alpha^a = \frac{\tau}{Z_t} \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^m \frac{t_{\lambda_\beta}^a Z_t}{x_\beta - x_\alpha} dq_\alpha. \quad (4.16)$$

### 4.3 Main claim

To summarize, we have constructed the multiple SLEs for  $\hat{\mathfrak{g}}_k$ -WZW models:

$$\begin{aligned} dg_t(z) &= \sum_{\alpha=1}^m \frac{2dq_\alpha}{g_t(z) - x_{\alpha t}}, & dx_{\alpha t} &= \sqrt{\kappa} d\xi_{\alpha t} + dF_{\alpha t} \\ d\theta_t^a(z) &= \sum_{\alpha=1}^m \frac{dp_{\alpha t}^a}{z - x_{\alpha t}}, & dp_{\alpha t}^a &= \sqrt{\tau} d\vartheta_{\alpha t}^a + dG_{\alpha t}^a \quad (1 \leq a \leq \dim(\mathfrak{g})). \end{aligned} \quad (4.17)$$

The drift terms  $dF_{\alpha t}$  and  $dG_{\alpha t}^a$  are given by

$$dF_{\alpha t} = \kappa dq_\alpha \partial_{x_{\alpha t}} \log Z_t + 2 \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^m \frac{dq_\beta}{x_{\alpha t} - x_{\beta t}}, \quad dG_{\alpha t}^a = \frac{\tau}{Z_t} \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^m \frac{t_{\lambda_\beta}^a Z_t}{x_{\beta t} - x_{\alpha t}} dq_\alpha, \quad (4.18)$$

where  $\xi_{\alpha t}$  (resp.  $\vartheta_{\alpha t}^a$ ) is the  $\mathbb{R}^m$  (resp.  $\mathbb{R}^{m \dim \mathfrak{g}}$ )-valued Brownian motion whose expectation value and variance are given by

$$\begin{aligned} \mathbf{E}[d\xi_{\alpha t}] &= 0, & \mathbf{E}[d\xi_{\alpha t} d\xi_{\beta t}] &= \delta_{\alpha\beta} dq_\alpha, \\ \mathbf{E}[d\vartheta_{\alpha t}^a] &= 0, & \mathbf{E}[d\vartheta_{\alpha t}^a d\vartheta_{\beta t}^b] &= \delta^{ab} \delta_{\alpha\beta} dq_\alpha. \end{aligned} \quad (4.19)$$

$Z_t$  is a partition function involving the bcc operators:

$$Z_t = \langle \psi_{\lambda_1}(x_{1t}) \cdots \psi_{\lambda_m}(x_{mt}) \psi_{\lambda_{m+1}}(\infty) \rangle, \quad (4.20)$$

where all the bcc operators  $\psi_{\lambda_\alpha}(x_\alpha)$  except for  $\psi_{\lambda_{m+1}}(\infty)$  must take values in representations specified by  $\lambda$ , or its conjugate  $\lambda^*$ . The conformal weight  $h_{\lambda_{m+1}}$  for the bcc operator inserted at  $\infty$  is determined by fusion rules (see next section for example). The structure of the partition function is described by both the global  $G$ -invariance (cf. (2.21)), and the KZ-equation (cf. (2.23))

$$\sum_{a=1}^{\dim \mathfrak{g}} \sum_{\alpha=1}^{m+1} t_{\lambda_\alpha}^a Z_t = 0, \quad \left( \partial_{z_\alpha} - \frac{1}{k + h^\vee} \sum_{a=1}^{\dim \mathfrak{g}} \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^m \frac{t_{\lambda_\alpha}^a t_{\lambda_\beta}^a}{z_\alpha - z_\beta} \right) Z_t = 0. \quad (4.21)$$

## 5 Multiple SLEs for $\widehat{\mathfrak{su}}(2)_k$ -WZW models

As a concrete application of our formula, let us consider multiple SLEs for the  $\widehat{\mathfrak{su}}(2)_k$ -WZW model where the bcc operators carry spin-1/2, and discuss topologies for the SLE traces.

For the  $\widehat{\mathfrak{su}}(2)_k$  case, the fusion rules of the primary fields are similar to those in the minimal CFTs. Therefore it is natural to extend the argument [17] describing the geometric configurations of the SLE traces in minimal CFTs to the  $\widehat{\mathfrak{su}}(2)_k$ -WZW model. Thus we make the following conjecture.

**Conjecture 5.1** *There exists a one-to-one correspondence between topologically inequivalent configurations of  $\widehat{\mathfrak{su}}(2)_k$  multiple SLE traces and the independent solutions of the KZ-equation satisfied by the partition functions.*

We describe this conjecture more specifically. Let  $x_\alpha$  ( $1 \leq \alpha \leq m$ ) be the positions where the  $m$  SLE traces start to grow, and be ordered as  $x_1 < \cdots < x_m < \infty$ . Consider the case that the traces eventually form  $m - n$  disjoint curves in  $\mathbb{H}$  so that each point  $x_\alpha$  is an end point of exactly one curve and  $\infty$  is an end point of exactly  $m - 2n \geq 0$  curves. Namely  $n$  disjoint curves form arches and other  $m - 2n$  curves converge toward the point at  $\infty$  (see Fig. 4 for  $m = 4$  and  $n = 2$ ). Then the number of topologically inequivalent configurations are given by

$$c_{m,n} = \binom{m}{n} - \binom{m}{n-1}. \quad (5.1)$$

This is nothing but a Kostka number appearing as the coefficient of the irreducible decomposition for the  $m$  tensor product of the  $\mathfrak{su}(2)$  fundamental representation  $L_\Lambda^{\otimes m}$  into  $L_{(m-2n)\Lambda}$  ( $L_{\lambda\Lambda}$  denotes the integral representation with the highest weight  $\lambda\Lambda$ ):

$$L_\Lambda^{\otimes m} = \bigoplus_{n=0}^{\lfloor m/2 \rfloor} c_{m,n} L_{m-2n}, \quad (5.2)$$

where  $\lfloor x \rfloor$  denotes the integer part of  $x$ .

Now we mention that the relation between the geometric configurations and the CFT partition functions  $Z$  (4.20) (hereafter we omit the index  $t$  to simplify the notation). To this end, we consider the fusions of the  $\widehat{\mathfrak{su}}(2)_k$  primary fields:

$$\psi_{\lambda\Lambda} \otimes \psi_{\mu\Lambda} = \bigoplus_{\substack{\nu=|\lambda-\mu| \\ \lambda+\mu+\nu \equiv 0 \pmod{2}}}^{\min(\lambda+\mu, 2k-\lambda-\mu)} \psi_{\nu\Lambda}, \quad (5.3)$$

where  $\psi_{\lambda\Lambda}$  stands for the  $\widehat{\mathfrak{su}}(2)_k$  bcc primary field taking values in the integral representation  $L_{\lambda\Lambda}$ . For sufficiently large  $k$ , the above rules reduce to the decomposition (5.2). Applying the fusion procedures (5.3) to the bcc operators  $\psi_{\lambda_\alpha}$  ( $1 \leq \alpha \leq m$ ) recursively, one can reduce  $Z$  to a two point function involving the fusion operator and the bcc operator at  $\infty$ . Thus, for the non-vanishing partition functions, the conformal weight  $h_{\lambda_{m+1}}$  of the bcc operator  $\psi_{\lambda_{m+1}}(\infty)$  must be equivalent to that of the fusion operator. Fig. 5 shows the possible conformal weight  $h_{\lambda_{m+1}}$  up to  $m = 4$ . Due to the fusion procedures (5.3), for sufficiently large  $k$ , one finds that the number of the paths from the left to  $h_{(m-2n)\Lambda}$  corresponds to the number of topologically

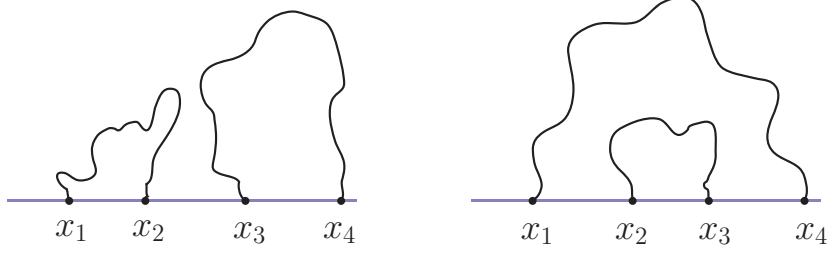


Figure 4: Configurations for  $m = 4$  and  $n = 2$ . There are two ( $c_{4,2} = \binom{4}{2} - \binom{4}{1} = 2$ ) topologically inequivalent configurations.

distinct configurations for the SLE traces, i.e.  $c_{m,n}$ . For generic  $k$ , the number of paths is, in general, constrained by the fusion rules, which affects the structure of the partition functions, and therefore the realization of the geometric configurations.

For instance, the configuration where all the curves eventually converge toward  $\infty$  (i.e. no arches) corresponds to the path to the weight  $h_{\lambda_{m+1}} = h_{m\Lambda}$ . By a standard argument for the CFT correlation functions, one obtains the corresponding partition function  $Z$ :

$$Z = \prod_{j < k}^m (x_j - x_k)^{\frac{1}{2(k+2)}} \quad \text{for } m \leq k. \quad (5.4)$$

One can easily check that this partition function satisfies the KZ-equation (4.21). The factorized correlation function with the same exponents  $1/(2(k+2))$  does not exist for  $m > k$ , indicating that the no-arch configuration is allowed only when  $m \leq k$ .

### 5.1 Double SLEs

According to the argument in [17], we confirm our conjecture by considering the simplest case ( $m = 2$ ), where only two SLE traces exist in the upper half plane  $\mathbb{H}$ . To analyze specifically, let us write down the partition function  $Z$ . Up to a constant factor, it is given by

$$Z = \langle \psi_\lambda(\infty) \psi_\Lambda(x_1) \psi_\Lambda(x_2) \rangle = \lim_{x \rightarrow \infty} x^{2h_\lambda} \langle \psi_\lambda(x) \psi_\Lambda(x_1) \psi_\Lambda(x_2) \rangle = (x_1 - x_2)^\Delta, \quad (5.5)$$

where the exponent is  $\Delta = h_{\lambda_3} - 2h_\Lambda$ . The fusion rules (5.3) (see also Fig. 5) indicate that  $h_{\lambda_3} = h_{2\Lambda}$  ( $k > 1$ ) or  $h_{\lambda_3} = 0$  ( $k \geq 1$ ). Then using the relation (4.12), one arrives at  $\Delta = 1/(2(k+2))$  ( $k > 1$ ) or  $\Delta = -3/(2(k+2))$  ( $k \geq 1$ ). Correspondingly the partition functions are

$$Z_0 = (x_1 - x_2)^{-\frac{3}{2(k+2)}} \quad (k \geq 1), \quad Z_2 = (x_1 - x_2)^{\frac{1}{2(k+2)}} \quad (k > 1). \quad (5.6)$$

By inserting them into (4.17) and (4.18), the driving processes  $dx_{1t}$  and  $dx_{2t}$  characterizing the geometric aspects of the SLE traces become:

$$dx_{1t} = \sqrt{a_1 \kappa} dB_{1t} + \frac{2a_2 + \kappa \Delta a_1}{x_{1t} - x_{2t}} dt, \quad dx_{2t} = \sqrt{a_2 \kappa} dB_{2t} + \frac{2a_1 + \kappa \Delta a_2}{x_{2t} - x_{1t}} dt, \quad (5.7)$$

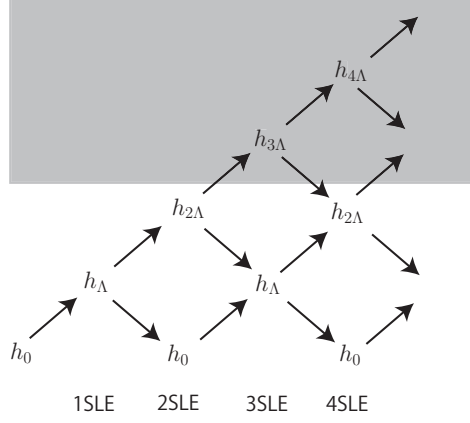


Figure 5: The fusion rules of the bcc operator  $\psi_\Lambda$ . The primary fields with weight  $h_{\lambda\Lambda}$  for  $\lambda > k$  do not exist due to the fusion rules. For instance, for  $k = 2$ , the primary fields in the shadow area vanish.

where we normalized the variances by  $dq_{\alpha t} = a_\alpha dt$  so that  $d\xi_{\alpha t} = \sqrt{a_\alpha} dB_{\alpha t}$  with two independent standard Brownian motions:  $\mathbf{E}[dB_{\alpha t}] = 0$  and  $\mathbf{E}[dB_{\alpha t} dB_{\beta t}] = \delta_{\alpha\beta} dt$ . These driving processes with the SLE (4.17) describe two curves emerging from two points  $x_1 = x_{1,0}$  and  $x_{2,0}$ . Setting  $y_s = x_{1t} - x_{2t}$  and rescaling the time by  $ds = \kappa(a_1 + a_2)dt$ , we reduce the processes to the following Bessel process:

$$dy_s = dB_s + \frac{\Delta + 2/\kappa}{y_s} ds \quad (5.8)$$

with the effective dimension  $d_{\text{eff}} = 2\Delta + 4/\kappa + 1$ . Substitution of the exponent  $\Delta$  and  $\kappa$  (4.13) yields

$$d_{\text{eff}} = \begin{cases} 1 & (k = 1) \\ \frac{2(k+1)}{k+2} & (k > 1) \end{cases} \quad \text{for } h_{\lambda_3} = 0, \quad d_{\text{eff}} = 2 + \frac{2}{k+2} \quad (k > 1) \quad \text{for } h_{\lambda_3} = h_{2\Lambda}. \quad (5.9)$$

Recalling that the Bessel process is recurrent (resp. not recurrent) if  $d_{\text{eff}} < 2$  (resp.  $d_{\text{eff}} > 2$ ), we conclude that the driving processes  $x_{\alpha t}$  hit each other with probability 1 for  $h_{\lambda_3} = 0$  and never hit for  $h_{\lambda_3} = h_{2\Lambda}$ . The hit of the driving processes means the hit of the tips of the SLE traces, and hence the case for  $h_{\lambda_3} = 0$  describes a single curve (i.e. single arch) whose end points are  $x_1$  and  $x_2$ , while the case for  $h_{\lambda_3} = h_{2\Lambda}$  describes two curves converging to the point at  $\infty$ . The above observation agrees with Conjecture 5.1.

## 5.2 Triple SLEs and arch (crossing) probabilities

Let us discuss a more non-trivial example: the triple SLEs ( $m = 3$ ). The geometric properties are characterized by the partition function:

$$Z = \langle \psi_{\lambda_4}(x_4) \psi_{\lambda_3}(x_3) \psi_{\lambda_2}(x_2) \psi_{\lambda_1}(x_1) \rangle, \quad (5.10)$$

where  $x_4 = \infty$  and by construction the bcc operators  $\psi_{\lambda_j}$  ( $1 \leq j \leq m$ ) take values in the fundamental representation  $L_\Lambda$ . From the fusion rules (see also Fig. 5), the conformal weight



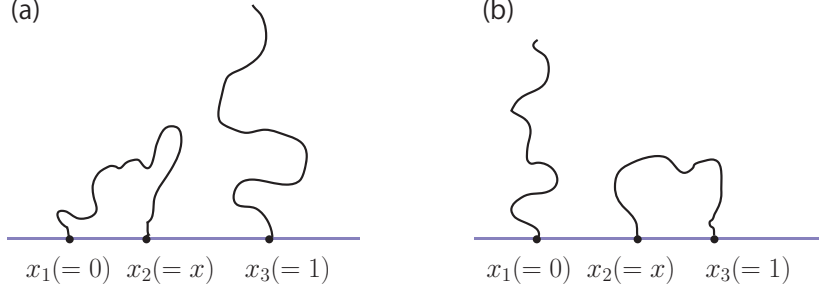


Figure 6: Configurations for  $m = 3$  and  $n = 1$ . There are two ( $c_{3,1} = \binom{3}{1} - \binom{3}{0} = 2$ ) topologically inequivalent configurations. These configurations are described by the partition function involving the 4 bcc operators with the same conformal weights  $h_\Lambda$ . The configuration in the left (resp. right) panel is characterized by the partition function  $Z_{C_1}$  (resp.  $Z_{C_2}$ ).

$h_{\lambda_4}$  of the bcc operator  $\psi_{\lambda_4}(\infty)$  inserted at  $\infty$  is  $h_{\lambda_4} = h_{3\Lambda}$  ( $k \geq 3$ ) or  $h_{\lambda_4} = h_\Lambda$  ( $k \geq 1$ ). When the level takes its value in the range  $k \geq 3$  and  $h_{\lambda_4} = h_{3\Lambda}$ , there exists the factorized partition function with the same exponents  $1/(2(k+2))$ :

$$Z = [(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)]^{\frac{1}{2(k+2)}} \quad (k \geq 3). \quad (5.11)$$

According to Conjecture 5.1, we expect that this partition function describes three curves converging toward  $\infty$ .

The case for  $h_{\lambda_4} = h_\Lambda$  is more interesting, because two topologically inequivalent configurations do exist (see Fig. 6). By conformal mapping  $f(z) = (z-x_1)(x_3-x_4)/((z-x_4)(x_3-x_1))$ , the four points  $x_j$  are transformed to the points  $x_1 \rightarrow 0$ ,  $x_2 \rightarrow x$ ,  $x_3 \rightarrow 1$  and  $x_4 \rightarrow \infty$ , where  $x = f(x_2)$ . Thus the partition function  $Z$  can be expressed as

$$Z = ((x_2 - x_4)(x_1 - x_3))^{-2h_\Lambda} Z(x), \quad Z(x) = \langle \psi_\Lambda(\infty) \psi_\Lambda(1) \psi_\Lambda(x) \psi_\Lambda(0) \rangle. \quad (5.12)$$

The correlation function  $Z(x)$  can be calculated by solving the KZ-equation (4.21) [22, 19]. Its explicit form reads

$$Z(x) = Z_{C_1}(x) + Z_{C_2}(x) \quad (5.13)$$

with

$$Z_{C_1}(x) = F_1^{(-)} + \frac{1-c_-}{c_+} F_1^{(+)}, \quad Z_{C_2}(x) = F_2^{(-)} + \frac{1-c_-}{c_+} F_2^{(+)}, \quad (5.14)$$

where  $F_j^{(\pm)}$ 's are expressed by the hypergeometric function  ${}_2F_1$ :

$$\begin{aligned} F_1^{(-)} &= x^{-2h_\Lambda} (1-x)^{h_{2\Lambda}-2h_\Lambda} {}_2F_1 \left( \frac{1}{k+2}, \frac{-1}{k+2}; \frac{k}{k+2}; x \right), \\ F_1^{(+)} &= x^{h_{2\Lambda}-2h_\Lambda} (1-x)^{h_{2\Lambda}-2h_\Lambda} {}_2F_1 \left( \frac{1}{k+2}, \frac{3}{k+2}; \frac{k+4}{k+2}; x \right), \\ F_2^{(-)} &= \frac{1}{k} x^{1-2h_\Lambda} (1-x)^{h_{2\Lambda}-2h_\Lambda} {}_2F_1 \left( \frac{k+3}{k+2}, \frac{k+1}{k+2}; 2\frac{k+1}{k+2}; x \right), \\ F_2^{(+)} &= -2x^{h_{2\Lambda}-2h_\Lambda} (1-x)^{h_{2\Lambda}-2h_\Lambda} {}_2F_1 \left( \frac{1}{k+2}, \frac{3}{k+2}; \frac{2}{k+2}; x \right), \end{aligned} \quad (5.15)$$

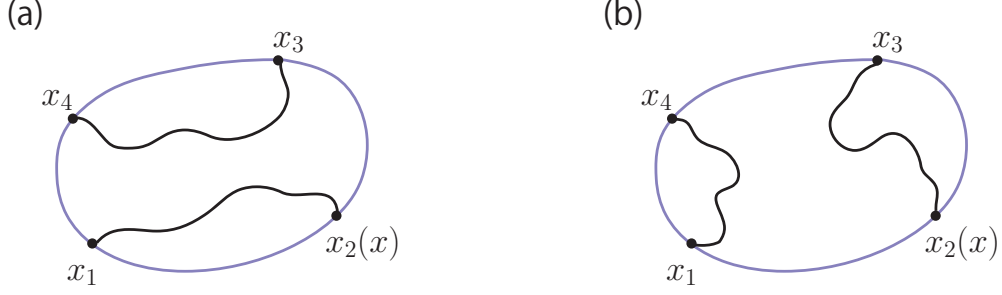


Figure 7: Arch configurations in arbitrary domain  $\mathcal{D}$  obtained by conformal transformation of the configurations in Fig. 6. The probability of the occurrence of the configuration (a) (resp. (b)) is exactly the same as that of the occurrence of the configuration (a) (resp. (b)) in Fig. 6.

and the coefficients  $c_{\pm}$  are respectively, given by

$$c_- = 2 \frac{\Gamma(2/(k+2))\Gamma(-2/(k+2))}{\Gamma(1/(k+2))\Gamma(-1/(k+2))}, \quad c_+ = -2 \frac{\Gamma^2(2/(k+2))}{\Gamma(3/(k+2))\Gamma(1/(k+2))}. \quad (5.16)$$

Note that  $Z_{C_1}(x)$  and  $Z_{C_2}(x)$  satisfy the relation  $Z_{C_1}(x) = Z_{C_2}(1-x)$ . For  $k=1$ ,  $c_- = 1$  holds<sup>3</sup>, and then the partition functions reduce to simple forms

$$Z_{C_1}(x) = x^{-1/2}(1-x)^{1/2}, \quad Z_{C_2}(x) = x^{1/2}(1-x)^{-1/2}. \quad (5.17)$$

For generic  $k$ , the behaviors of the partition functions close to the position  $x=0$  and  $x=1$  are described as

$$Z_{C_1}(x) \sim \begin{cases} x^{-2h_{\Lambda}} & x \rightarrow 0 \\ (1-x)^{h_{2\Lambda}-2h_{\Lambda}} & x \rightarrow 1 \end{cases}, \quad Z_{C_2}(x) \sim \begin{cases} x^{h_{2\Lambda}-2h_{\Lambda}} & x \rightarrow 0 \\ (1-x)^{-2h_{\Lambda}} & x \rightarrow 1 \end{cases}. \quad (5.18)$$

Thus the fusion rules (5.3) and Conjecture 5.1 imply that  $Z_{C_1}$  describes two curves connecting the points  $[0x]$  and  $[1\infty]$  (configuration  $C_1$ ; see the left panel in Fig 6) while  $Z_{C_2}$  corresponds to the configuration  $[\infty 0]$  and  $[x1]$  (configuration  $C_2$ ; see the right panel in Fig 6).

Utilizing the partition functions (5.14), we can exactly compute the probability  $\mathbf{P}[C_1]$  (resp.  $\mathbf{P}[C_2]$ ) of the occurrence of the configuration  $C_1$  (resp.  $C_2$ ) with the initial condition that the three curves emerging from the point  $x_1=0$ ,  $x_2=x$ ,  $x_3=1$  by

$$\mathbf{P}[C_1] = \frac{Z_{C_1}(x)}{Z_{C_1}(x) + Z_{C_2}(x)}, \quad \mathbf{P}[C_2] = \frac{Z_{C_2}(x)}{Z_{C_1}(x) + Z_{C_2}(x)}. \quad (5.19)$$

Since the measure is conformally invariant, the probabilities in arbitrary (simply connected) domain  $\mathcal{D}$  can be easily reproduced by use of  $Z(x)$  and the conformal map  $f^{-1}(z)$ . More explicitly, the probability  $\mathbf{P}[C_1]$  (resp.  $\mathbf{P}[C_2]$ ) is exactly the same as that of the occurrence of the configuration with two curves joining the boundary points  $[x_1x_2(x)]$  and  $[x_3x_4]$  (resp.

<sup>3</sup>Thus the second terms in (5.14) vanish. In fact, this is a consequence that primary fields with weight  $h_{2\Lambda}$  do not exist for  $k=1$ .

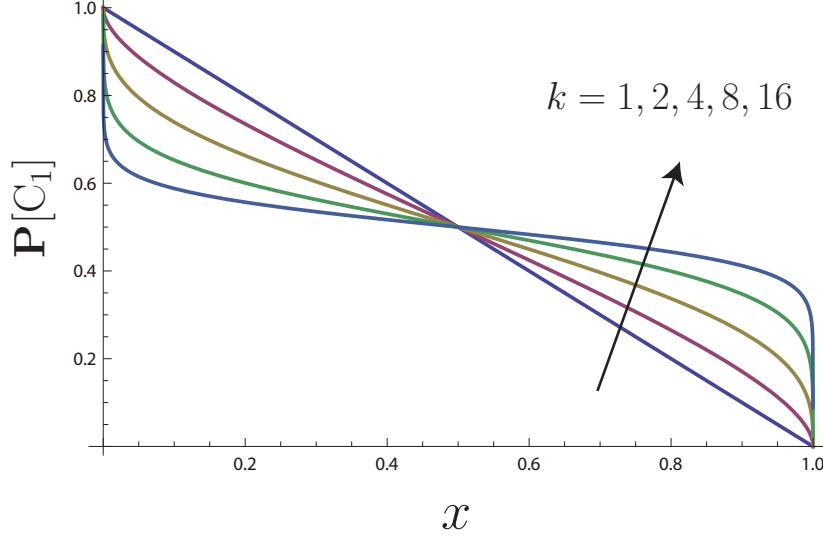


Figure 8: The crossing probabilities  $\mathbf{P}[C_1]$  for various level  $k$ .

$[x_1x_4]$  and  $[x_2(x)x_3]$ ) on  $\partial\mathcal{D}$  (see Fig. 7) (note that  $x_2$  depends on  $x$ , i.e.  $x_2(x) = f^{-1}(x)$ ). In this sense, the arch probabilities can be interpreted as crossing probabilities of the interfaces.

For  $k = 1$ , the formula becomes very simple:

$$\mathbf{P}[C_1] = 1 - x \quad (k = 1). \quad (5.20)$$

In Fig. 8, the arch probabilities  $\mathbf{P}[C_1]$  are depicted for several values of  $k$ . As shown in Fig. 8, the  $x$ -dependence of the arch probabilities close to  $1/2$  in the wide range of  $x$ , as increasing  $k$ .

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